Exponential and Logarithmic Functions

Finite Math

7 January 2019

Definition

Definition (Exponential Function)

An exponential function is a function of the form

$$f(x) = b^x, \; b > 0, \; b \neq 1.$$

b is called the base.

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  an imaginary number! This kind of thing will always happen if b is negative.
- If $b = 0$, then for negative x values, f is not defined. For example,

$$f(-1) = 0^{-1} = \frac{1}{0} = \text{undefined.}$$

Graphing Exponential Functions

Example

Sketch the graph of $f(x) = 2^x$.

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Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x$.

Negative Powers

Notice that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$$

so that when $b < 1$, we can set $b = \frac{1}{c}$ and have $c > 1$ and

$$f(x) = b^x = \left(\frac{1}{c}\right)^x = c^{-x}.$$

So, we can always keep the base larger than 1 by using a minus sign in the exponent if necessary.

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4. b^x *is increasing if* $b > 1$.
5. b^x *is decreasing if* $0 < b < 1$.

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3. $a^x = b^x$ *for all* x *if and only if* $a = b$

The Natural Number

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approaches as x tends towards ∞.
This number often shows up in growth and decay models, such as population growth, radioactive decay, and continuously compounded interest. If c is the initial amount of the measured quantity, and r is the growth/decay rate of the quantity ($r > 0$ is for growth, $r < 0$ is for decay), then the amount after time t is given by

$$A = ce^{rt}.$$

Growth and Decay Example

Example

In 2013, the estimated world population was 7.1 *billion people with a relative growth rate of* 1.1%.

(a) *Write a function modeling the world population* t *years after 2013.*

(b) *What is the expected population in 2015? 2025? 2035?*

Now You Try It!

Example

The population of some countries has a relative growth rate of 3% per year. Suppose the population of such a country in 2012 is 6.6 million.

(a) *Write a function modeling the population t years after 2012.*

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Solution

(a) $P = 6.6e^{0.03t}$

(b) 7.90 *million;* 8.91 *million*

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Consider the function $f(x) = x^2$. If we run f backwards on the value 1, what x-value do we get?
Since $(1)^2 = 1$ and $(-1)^2 = 1$, we get *two* values when we run x^2 backward! So x^2 is not invertible.

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If we have a one-to-one function

$$y = f(x)$$

we can form the *inverse function* by switching x and y and solving for y:

$$x = f(y) \xrightarrow{\text{solve for } y} y = f^{-1}(x).$$

Logarithms

We will focus on one particular inverse function: the inverse of the function $f(x) = b^x$ ($b > 0$, $b \neq 1$).

Definition (Logarithm)

The logarithm of base b is defined as the inverse of b^x. That is,

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8. $\log_b M = \log_b N$ *if and only if* $M = N$

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$$\log_b x = \frac{\ln x}{\ln b}$$

(See the textbook for a proof of this.)

Using Properties of Exponents and Logarithms

Example

Solve for x in the following equations:

(a) $7 = 2e^{0.2x}$

(b) $16 = 5^{3x}$

(c) $8000 = (x - 4)^3$

Reminder of Some Exponent Types

A quick reminder of different types of exponents:

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Now You Try It!

Example

Solve for x in the following equations:

(a) $75 = 25e^{-x}$

(b) $42 = 7^{2x+3}$

(c) $200 = (2x - 1)^5$

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Solution

(a) $x \approx -1.09861$

(b) $x \approx -0.53961$

(c) $x \approx 1.94270$

Applications

Recall that exponential growth/decay models are of the form

$$A = ce^{rt}.$$

Using the natural logarithm, we can solve for the rate of growth/decay, r, and the time elapsed, t. Let's see this in an example.

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Example

The isotope carbon-14 has a half-life (the time it takes for the isotope to decay to half of its original mass) of 5730 years.

(a) *At what rate does carbon-14 decay?*

(b) *How long would it take for 90% of a chunk of carbon-14 to decay?*